

# Stability of Stationary Solutions of Extended Reaction-Diffusion-Convection Equations on a Finite Segment

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**Abstract.** A simple geometric criterion on the linear stability of stationary solutions of nonlinear second order parabolic equations on a finite segment is stated and proved.

This note deals with the linear stability of the stationary solutions of the problem

$$f(u, u_x, u_{xx}, u_t) = 0 \text{ in } 0 < x < l, t > 0, \quad (1)$$

$$B_0(u, u_x) = 0 \text{ at } x = 0, \quad B_1(u, u_x) = 0 \text{ at } x = l, \quad (2)$$

where  $f$ ,  $B_0$  and  $B_1$  are  $C^1$  functions such that  $f_q(u, p, q, r) > 1$ ,  $f_r(u, p, q, r) < -1$  whenever  $f(u, p, q, r) = 0$  (i.e., equation (1) is uniformly parabolic), and, for  $i = 0, 1$ ,  $B_{iu}(u, p)^2 + B_{ip}(u, p)^2 \neq 0$  whenever  $B_i(u, p) = 0$  (i.e., the boundary conditions (2) define simple curves in the phase plane of (1) with  $u_t = 0$ ). If, in addition,  $f$  depends linearly on  $p$ ,  $q$  and  $r$ , then (1)–(2) is a standard reaction-diffusion-convection problem. See [1–2] and [3–4] for applications in the frameworks of charge distributions in semiconductors and of reactant concentration and temperature distributions in porous catalysts.

Linear stability properties of an stationary solution of (1)–(2),  $U$ , are defined in terms of the sign of the largest eigenvalue,  $\lambda_1$ , of the linearised problem

$$\mathcal{L}(U)v = \lambda v \text{ in } 0 < x < l, \quad \mathfrak{B}_i(U)v = 0 \text{ at } x = il, \text{ for } i = 0, 1, \quad (3)$$

where the operators  $\mathcal{L}(U)$ ,  $\mathfrak{B}_0(U)$  and  $\mathfrak{B}_1(U)$  are defined as

$$\mathcal{L}(U)v \equiv \phi(x)v'' + \varphi(x)v' + \psi(x)v, \quad \mathfrak{B}_i(U)v \equiv a_i(U)v' + b_i(U)v, \quad (4)$$

where, for  $x \in (0, l)$ ,  $\phi(x) = -f_q/f_r$ ,  $\varphi(x) = -f_p/f_r$ , and  $\psi(x) = -f_u/f_r$ , at  $(u, p, q, r) = (U(x), U_x(x), U_{xx}(x), 0)$ , and for  $i = 0, 1$ ,

$$a_i(U) = B_{ip}, \quad b_i(U) = B_{iu} \text{ at } (u, p) = (U(il), U_x(il)). \quad (5)$$

As is well known, the eigenvalues of (3) are real, and the eigenfunctions associated with  $\lambda_1$  do not vanish in  $0 < x < l$ . The stationary solution  $U$  is linearly exponentially stable (resp. linearly stable or unstable) if  $\lambda_1 < 0$  (resp.  $\lambda_1 \leq 0$  or  $\lambda_1 > 0$ ).

We introduce for convenience the following definitions concerning the boundary conditions. If  $(-1)^{i+1} a_i(U) U_x(il) [a_i(U) U_{xx}(il) + b_i(U) U_x(il)] < 0$  (resp.  $= 0$  or  $> 0$ ),  $a_i(U) \neq 0$  and  $U_x(il) \neq 0$ , then the boundary condition (2) at  $x = il$  is said to be of *type*  $A^-$  (resp.  $A^0$  or  $A^+$ ) *with respect to the stationary solution*  $U$ . If  $a_i(U) = 0$  and  $U_x(il) = 0$  (resp.  $\neq 0$ ) then the boundary condition (2) at  $x = il$  is of *type*  $A^0$  (resp.  $A^+$ ). Finally, if  $a_i(U) \neq 0$  and  $U_x(il) = 0$  then the boundary condition (2) at  $x = il$  is of *type*  $A^-$ .

Below,  $l$  will be treated as a bifurcation parameter, and the stationary solution  $U$  imbedded in a one-parameter family of stationary solutions of (1),  $(x, \mu) \rightarrow V(x, \mu)$ , as follows. For each (sufficiently small)  $\mu$ ,  $V(\cdot, \mu)$  is the unique solution of the initial value problem posed by

$$f(V, V_x, V_{xx}, 0) = 0 \text{ if } x > 0, \quad B_0(V, V_x) = 0 \text{ at } x = 0, \quad (6)$$

$$V = U(0) + \mu \text{ if } a_0(U) \neq 0, \quad V_x = U_x(0) + \mu \text{ if } a_0(U) = 0, \text{ at } x = 0. \quad (7)$$

Notice that  $V(x, 0) \equiv U(x)$ . If the boundary condition at  $x = l$  is not of type  $A^0$  with respect to  $U$ , i.e., if  $a_1(U)U_{xx}(l) + b_1(U)U_x(l) \neq 0$ , then, for sufficiently small  $\mu$ , a function  $\mu \rightarrow L(\mu)$  is well defined by

$$B_1(V(L(\mu), \mu), V_x(L(\mu), \mu)) \equiv 0, \quad L(0) = l, \quad (8)$$

as obtained when applying standard results on the regular dependence of the solution of initial value problems (such as (6)–(7)) on initial data, and the implicit function theorem. Also, by differentiating in (6)–(8) with respect to  $\mu$ , we find that the function  $v_1(x) \equiv V_\mu(x, 0)$  satisfies

$$\mathfrak{L}(U)v_1 = 0 \text{ in } 0 < x < l, \quad (9)$$

$$\mathfrak{B}_0(U)v_1 = 0, \quad v_1 = 1 \text{ if } a_0(U) \neq 0, \quad v'_1 = 1 \text{ if } a_0(U) = 0, \text{ at } x = 0, \quad (10)$$

$$\mathfrak{B}_1(U)(v_1 + l_\mu v_0) = 0 \text{ at } x = l, \text{ where } l_\mu \equiv L'(\mu) \text{ and } v_0 \equiv U_x. \quad (11)$$

To end up these preliminaries, we give the following well known Sturm comparison theorem, to be used systematically in the sequel. The result is a straightforward generalisation of the standard comparison theorem in [5, p. 208, Theorem 1.1], and its simple proof is included for the sake of completeness.

**LEMMA.** Let the operator  $\mathfrak{L}$  be defined by  $\mathfrak{L}v \equiv \phi(x)v'' + \varphi(x)v' + \psi(x)v$ , where the continuous functions  $\phi$ ,  $\varphi$  and  $\psi$  are such that  $\phi(x) > 1$  in  $l_1 < x < l_2$ , and let the  $C^2$ -functions  $w_1$  and  $w_2$  and the constants  $\sigma_1$  and  $\sigma_2$  be such that

$$\mathfrak{L}w_1 = \sigma_1 w_1, \quad \mathfrak{L}w_2 = \sigma_2 w_2 \text{ in } l_1 < x < l_2, \quad (12)$$

$$w'_1(l_1)w_2(l_1) \leq w_1(l_1)w'_2(l_1), \quad w'_1(l_2)w_2(l_2) \geq w_1(l_2)w'_2(l_2). \quad (13)$$

If  $w_1 w_2 > 0$  in  $l_1 < x < l_2$ , then  $\sigma_1 \geq \sigma_2$ . If, in addition, one of the inequalities (13) is strict, then  $\sigma_1 > \sigma_2$ .

**PROOF:** The result follows from the inequality

$$\int_{l_1}^{l_2} (\sigma_1 - \sigma_2) \phi(x)^{-1} w_1(x) w_2(x) \exp \left[ \int_{l_1}^x \varphi(\xi) \phi(\xi)^{-1} d\xi \right] dx \geq 0, \quad (14)$$

that is a strict inequality if one of the inequalities (13) is strict. Equation (14) is obtained, when taking into account (13), upon multiplication of the first equation in (12) by  $\phi(x)^{-1} w_2(x) \exp[\int_{l_1}^x \varphi(\xi) \phi(\xi)^{-1} d\xi]$ , multiplication of the second equation in (12) by  $\phi(x)^{-1} w_1(x) \exp[\int_{l_1}^x \varphi(\xi) \phi(\xi)^{-1} d\xi]$ , subtraction, integration in  $(0, l)$ , and integration by parts.

The main result of this note is contained in the following

**THEOREM.** Under the assumptions above, let  $U$  be a stationary solution of (1)–(2), let  $\lambda_1$  be the largest eigenvalue of the linearised problem (3), let  $n$  be the number of critical points of  $U$  in the interval  $(0, l)$ , and let  $X$  and  $Y$  ( $= A^-, A^0$  or  $A^+$ ) be the type of boundary conditions (2), at  $x = 0$  and  $x = l$  respectively, with respect to  $U$ . Then:

- (1)  $\lambda_1 < 0$  if  $n = 0$  and either (i)  $X = A^+$  and  $Y = A^0$  or  $A^+$ , or (ii)  $X = A^+$  or  $A^0$  and  $Y = A^+$ .

- (2)  $\lambda_1 = 0$  if  $n = 0$  and  $X = Y = A^0$ .
- (3)  $\lambda_1 > 0$  if either (i)  $n \geq 2$ , or (ii)  $n = 1$  and either  $X$  or  $Y$  is equal to  $A^-$  or  $A^0$ , or (iii)  $n = 0$ ,  $X = A^-$  and  $Y = A^-$  or  $A^0$ , or (iv)  $n = 0$ ,  $X = A^-$  or  $A^0$  and  $Y = A^-$ .
- (4)  $\text{sgn}(\lambda_1) = -\text{sgn}(U_x(0)l_\mu)$  if either (i)  $n = 1$  and  $X = Y = A^+$ , or (ii)  $n = 0$ ,  $X = A^+$  and  $Y = A^-$ .
- (5)  $\text{sgn}(\lambda_1) = \text{sgn}(U_x(l)l_\mu)$  if  $n = 0$ ,  $X = A^-$  and  $Y = A^+$ .

Here  $\text{sgn}(x) = x/|x|$  if  $x \neq 0$ ,  $\text{sgn}(0) = 0$  and  $l_\mu$  is defined in (11).

PROOF: Let  $v (> 0$  in  $0 < x < l$ ) be an eigenfunction associated with  $\lambda_1$ , and let  $v_0 \equiv U_x$ . Notice that  $\mathfrak{L}(U)v_0 = 0$  in  $0 < x < l$ .

- (1) Apply the Lemma with  $w_1 = |v_0|$ ,  $\sigma_1 = 0$ ,  $w_2 = v$ ,  $\sigma_2 = \lambda_1$ ,  $l_1 = 0$  and  $l_2 = l$ .
- (2) As in the previous case, it is seen that  $\lambda_1 \leq 0$ . Also,  $\lambda = 0$  is an eigenvalue of (3) ( $v_0$  is the associated eigenfunction).
- (3) Apply the Lemma with  $w_1 = v$ ,  $\sigma_1 = \lambda_1$ ,  $w_2 = |v_0|$ ,  $\sigma_2 = 0$  and: (i)  $l_1$  and  $l_2$  equal to two consecutive zeroes of  $v_0$ ; (ii)  $l_1 = 0$  (resp.  $l_2 = l$ ) and  $l_2$  (resp.  $l_1$ ) equal to the zero of  $v_0$  in  $(0, l)$  if  $X = A^-$  or  $A^0$  (resp. if  $Y = A^-$  or  $A^0$ ); (iii)–(iv)  $l_1 = 0$  and  $l_2 = l$ .
- (4) If (i) holds, then  $v_0(0)v_0(l) < 0$  and (see (11))

$$\begin{aligned} & \text{either } a_1(U) = 0 \text{ and } \text{sgn}(v_1(l)) = \text{sgn}(l_\mu v_0(0)), \\ & \text{or } a_1(U) \neq 0 \text{ and } \text{sgn}[a_1(U) \mathfrak{B}_1 v_1] = \text{sgn}(l_\mu v_0(0)). \end{aligned} \quad (15)$$

Then  $v_1 > 0$  in  $0 < x \leq x_0$ , and either  $v_1 > 0$  in  $x_0 < x < l$  or  $v_1$  has a simple zero in  $(x_0, l)$ , where  $x_0$  is the zero of  $v_0$  in  $(0, l)$ . This is proven by a contradiction argument. Assume, on the contrary, that either (a)  $0 < x_1 \leq x_0$  or (b)  $x_0 < x_1 < x_2 < l$ , for some  $x_1$  and  $x_2$  such that  $v_1(x_1) = v_1(x_2) = 0$ ,  $v_1 > 0$  in  $(0, x_1)$  and  $v_1 \neq 0$  in  $(x_1, x_2)$ . If (a) (resp. (b)) holds, then, by applying the Lemma with  $w_1 = |v_0|$ ,  $w_2 = |v_1|$ ,  $\sigma_1 = \sigma_2 = 0$ ,  $l_1 = 0$  and  $l_2 = x_1$  (resp.  $l_1 = x_1$  and  $l_2 = x_2$ ) we obtain  $0 = \sigma_1 < \sigma_2 = 0$  (thus, a contradiction).

Now, if  $l_\mu v_0(0) \geq 0$ , then  $v_1 > 0$  in  $0 < x < l$  (otherwise,  $x_1 \in (0, l)$  would exist such that  $v_0 \neq 0$  and  $v_1 < 0$  in  $x_1 < x < l$ ,  $v_0(x_1) \neq 0$  and  $v_1(x_1) = 0$ , and a contradiction would be obtained by applying the Lemma with  $w_1 = |v_0|$ ,  $w_2 = -v_1$ ,  $\sigma_1 = \sigma_2 = 0$ ,  $l_1 = x_1$  and  $l_2 = l$ ). Let  $l_\mu v_0(0) > 0$ . Application of the Lemma with  $w_1 = v_1$ ,  $w_2 = v$ ,  $\sigma_1 = 0$ ,  $\sigma_2 = \lambda_1$ ,  $l_1 = 0$  and  $l_2 = l$  yields  $\lambda_1 < 0$ . If  $l_\mu = 0$  then  $\lambda = 0$  is an eigenvalue of (3), with  $v_1$  as an associated eigenfunction. On the other hand, if  $v_0(0)l_\mu < 0$  then either (a)  $v_1 > 0$  in  $0 < x < l$ , or (b)  $v_1 > 0$  in  $0 < x < x_1 < l$  and  $v_1(x_1) = 0$ . If (a) (resp. (b)) holds, then the result  $\lambda_1 > 0$  follows from the Lemma with  $w_1 = v$ ,  $w_2 = v_1$ ,  $\sigma_1 = \lambda_1$ ,  $\sigma_2 = 0$ ,  $l_1 = 0$  and  $l_2 = l$  (resp.  $l_2 = x_1$ ).

If (ii) holds, then  $v_0(0) \neq 0$ ,  $a_1(U) \neq 0$  and either  $v_0(l) = 0$  and  $v'_0(l)v_0(0) < 0$ , or  $v_0(l)v_0(0) > 0$ . Then, the second alternative (15) holds and, by the argument above,  $v_1 > 0$  in  $0 < x < l$ . The proof proceeds as in case (i).

- (5). The proof is similar to that of part (4), case (ii).

REMARK 1. The type ( $A^-$ ,  $A^0$  or  $A^+$ ) of the boundary conditions is readily obtained once  $U$  (or  $U_x$ ) is known at  $x = 0$  or  $l$ . This type has an obvious interpretation in the phase plane of (1) with  $u_t = 0$ . The sign of  $l_\mu$  is connected with the slope of the bifurcation diagram of (1)–(2), with  $U(0)$  or  $U_x(0)$  as a “norm” to describe  $U$ , and  $l$  as a bifurcation parameter.

REMARK 2. All possible values of  $n$ ,  $X$  and  $Y$  are covered in parts (1)–(5) of the theorem.

REMARK 3. Equivalent results were obtained by Jetschke [6] for semi-linear equations ( $u_t = u_{xx} + F(u)$ ) with Dirichlet boundary conditions.

REMARK 4. Infinite domains are treated similarly. Consider  $l = \infty$ ,  $f(0, 0, 0, 0) = 0$  and the second boundary condition in (2) is replaced by imposing that  $u(\cdot, t) \in L_2(0, \infty)$ , or  $u(\cdot, t) \in C_{\text{unif}}[0, \infty)$  (= the space of uniformly continuous functions in  $[0, \infty)$ , with the sup norm) for all  $t > 0$ . Let  $n$  and  $X$  be defined as above, let  $\alpha = -f_u/f_r$  at  $u = p = q = r = 0$ , and let  $\lambda_1$  be the maximum of the spectrum of (3) (with  $U$  a stationary solution such that  $U(\infty) = 0$  and the second boundary condition replaced by  $v \in L_2(0, \infty)$ , or  $v \in C_{\text{unif}}[0, \infty)$ ).

Then  $\lambda_1 < 0$  if  $n = 0$ ,  $\alpha < 0$  and  $X = A^+$ ;  $\lambda_1 = 0$  if  $n = 0$  and either  $\alpha = 0$  and  $X = A^+$  or  $\alpha \leq 0$  and  $X = A^0$ ; and  $\lambda_1 > 0$  if either  $n \geq 1$ , or  $\alpha > 0$ , or  $X = A^-$ . This is proved by using the ideas above and some spectral theory (from, e.g., [7]) to deal with the essential spectrum, as in [8, Ch. 5, Appendix]. If Eq. (1) is considered in  $-\infty < x < \infty$ , then linear stability properties of the stationary solutions are completely analysed with the ideas in, e.g., [8, Sect. 5.4, p. 128]. Global stability properties are considered in [9].

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